Received May 13, 1773.

XXXVI. *Properties of the Conic Sections; deduced by a compendious Method. Being a Work of the late* William Jones, *Eſq; F. R. S. which he formerly communicated to Mr.* John Robertſon, *Libr. R. S. who now addreſſes it to the Reverend* Nevil Maſkelyne, *F. R. S. Aſtronomer Royal.*

SIR,

Read June 24, 1773.

YOU well know that the curves formed by the ſections of a cone, and therefore called Conic Sections, have, from the earlieſt ages of geometry, engaged the attention of mathematicians, on account of their extenſive utility in the ſolution of many problems, which were incapable of being conſtructed by any poſſible combination of right lines and circles, the magnitudes uſed in plane geometry. The properties of theſe curves are become far more intereſting within the two laſt centuries, ſince they have been found to be ſimilar to thoſe which are deſcribed by the motions of the cœleſtial bodies in the Solar ſyſtem.

Two different methods have been taken by the writers who have treated of their properties; the one, and the more antient, is to deduce them from the properties of the cone itſelf; the other is to conſider the curves, as generated by the conſtant motion of two or more ſtrait lines moving in a given plane, by certain laws.

There are various methods of generating theſe curve lines *in plano*; one method will give ſome properties very eaſily; but others, with much trouble: while, by another mode of deſcription, ſome properties may be readily derived, which, by the former, were not ſo eaſily come at: ſo that it appears there may be a manner of deſcribing the curves ſimilar to the Conic Sections, by the motion of lines on a plane, which, in general, ſhall produce the moſt eſſential properties, with the greateſt facility.

That excellent mathematician, the late William Jones, Eſq; F. R. S. had drawn up ſome papers on the deſcription of theſe curves, or lines of the ſecond kind, very different from what he gave in his *Synopſis Palmariorum Matheſeos*, publiſhed in the year 1706; or from that of any other writer on this ſubject. A copy of theſe papers he was pleaſed to let me take about the year 1740. He had not finiſhed them as he intended; but, in their preſent ſtate, they appear of too much conſequence to be loſt; as, it is much to be feared, his own copy, together with many other valuable papers, are; and therefore, I am deſirous of preſerving them in the Philoſophical Tranſ-

Tranſactions, in the manner I at firſt tranſcribed them; although, I am aware, they might have been put into a form more pleaſing to the generality of readers: I have indeed annexed larger diagrams than what accompanied the author's copy, in order to render the lines more diſtinct, as all the relations are to be repreſented in a ſingle figure, of each kind.

Mr. Jones, having laid down a very ſimple method of deſcribing theſe curves, ſeems to have been deſirous of arriving at their properties in as expeditious a way as he could contrive; and therefore he has uſed the algebraic method, in general, of reducing his equations; and, on ſome occaſions, has uſed the method of fluxions, to deduce ſome properties chiefly relating to the tangents; and, by a judicious uſe of theſe, he has very much abridged the ſteps which otherwiſe he muſt have taken, to have deduced the very great variety of relations he has obtained: theſe he intended to have arranged in tables, from whence an equation expreſſing the relation between any three or more lines of the Conic Sections, might be taken out as readily as a logarithm out of their tables; this he has only partly executed; but it may eaſily be continued by thoſe who are deſirous to have it done, and are ſufficiently acquainted with what follows.

From the Houſe of the Royal Society,
April 29, 1773.

THE DESCRIPTION OF LINES OF THE SECOND KIND.

LET the right lines AD, AQ, be drawn on a plane, at any inclination the one with the other. See PLATE XIV. Fig. 1, 2, 3.

In AD, AQ, take A*a*, AM, of any given magnitude, and draw MN parallel to AD,

On the points A, *a*, let two rulers AP, *a*P, revolve, and cut MN, AQ, in N and Q, ſo that AQ be every-where equal to MN.

Then ſhall the interſection P of the rulers deſcribe lines of the ſecond kind, or curves of the firſt kind.

Where the right-line A*a*, is the firſt, or tranſverſe diameter.

The point C, biſecting the diameter A*a*, is the center.

The right-line PD, drawn parallel to AQ, is the ordinate to the diameter A*a*.

The part AD, or CD, of the diameter, is the abſciſs, when reckoned to begin from A to C, or from C to A.

The right line B*b* drawn from the center C parallel to the ordinate PD, and terminated in the curve, is called the ſecond, or conjugate diameter.

Thoſe diameters to which the ordinates are perpendicular, are called the axes.

And AM is the parameter to the diameter A*a*.

THE PROPERTIES OF LINES OF THE SECOND KIND.

1. Put $\mathrm{A}a = d = 2\mathrm{AC} = 2t$; $\mathrm{B}b = \delta = 2\mathrm{BC} = 2c$; $\mathrm{AM} = 2p$; $\mathrm{PD} = y$; $\mathrm{CD} = x$; $\mathrm{AD} = u$. Then $\overline{\mathrm{PD}}^2 = \frac{p}{t} \times \mathrm{AD}a$.

Or $yy = \frac{p}{t} \times u \times \overline{d \mp u} = 2pu \mp \frac{p}{t} uu = \frac{p}{t} \times \overline{\pm\, tt \mp xx} = \pm pt \mp \frac{p}{t} x^2$.

For $\mathrm{PD} = \frac{\mathrm{AM} \times \mathrm{AD}}{\mathrm{MN}} = \frac{\mathrm{AQ} \times \mathrm{D}a}{\mathrm{A}a}$ (by ſim. △s). Th. $\overline{\mathrm{PD}}^2 = \frac{\mathrm{AM}}{\mathrm{A}a} \times \mathrm{AD}a$.

2. Conſequently $\frac{t}{p} yy = \pm tt \mp xx = du \mp uu = \pm \frac{1}{4} dd \mp xx$.

3. Hence $\frac{t}{p} y\dot{y} = \mp x\dot{x} = \overline{t \mp u} \times \dot{u} = x\dot{u}$.

4. And $pt = cc$, or $2pd = \delta\delta$; for when $y = c = \frac{1}{2}\delta$, then $x = 0$.

5. Therefore $\frac{\overline{\mathrm{PD}}^2}{\mathrm{AD}a} = \frac{yy}{\pm tt \mp xx} = \frac{p}{t} = \frac{2p}{d} = \frac{2pd}{dd} = \frac{pt}{tt} = \frac{\delta\delta}{dd} = \frac{cc}{tt} = \frac{\overline{\mathrm{BC}}^2}{\overline{\mathrm{AC}}^2}$.

6. The curve line whoſe property is $yy + \frac{p}{t} uu - 2pu = 0$,

(where the abſciſſa begins at the curve),

Or $yy + \frac{cc}{tt} xx - cc = 0$, (where it begins at the center),

is called an Ellipſis. This curve returns into itſelf. For when $x = 0$, then $y = c$; and when $y = 0$, then $x = t$. Which can happen but two ways.

7. The curve line whoſe property is $yy - \frac{p}{t} uu - 2pu = 0$,

Or $yy - \frac{cc}{tt} xx + cc = 0$, is called an Hyperbola. This curve ſpreads out infinitely. For y increaſes as x increaſes; and has four legs tending contrary ways: for xx, or yy, may be produced as well from $-x$, or $-y$; as from $+x$, or $+y$.

8. If the point a, is ſuppoſed to be at an infinite diſtance from A, ſo that a ruler $a\mathrm{P}$ moves in a parallel poſition to AD; then is $yy = 2pu$, or $yy - 2pu = 0$, the property of the curve deſcribed, and is called a Parabola. This curve ſpreads out infinitely; for y increaſes as u increaſes.

9. Let Aa, Pp, be any two firſt diameters; Bb, Qq, their ſecond diameters. Plate xiv. Fig. 4.

Draw the ordinates PD, QE, to the diameter Aa, and the ordinate Pd, to the diameter B b.

Let PT be a tangent, and PM be perpendicular, to the curve, in P; PT cutting Aa, Bb, produced, in T, t; and PM, in M, m.

Put the ſubtangent DT $= s$, $dt = \sigma$.

Let AT $= r$, PM $= \pi$, CE $= w$, CT $= q = x \pm s = t \pm r$.

Put s, s', = ſine and coſine of the angle MPD, or angle PmB.

R = tabular radius.

Then CDT $= xs = \overline{t \mp u} \times s = \frac{t}{p} yy = \frac{tt}{cc} yy = \pm\, tt \mp xx = u \times \overline{d \mp u} =$ ADa.

For $\dot{u} \times \overline{t \mp u} = \left(\frac{t}{p} y\dot{y} =\right) \alpha y \dot{y}$. Therefore $\left(\frac{\dot{u}}{\dot{y}} = \frac{\alpha y}{t \mp u} =\right) \frac{\alpha y}{x} = \frac{s}{y}$ (by ſim. $\triangle$s).

And C$dt = y\sigma = \frac{p}{t} xx = \frac{cc}{tt} xx = \pm cc \mp yy =$ Bdb. For $\left(\frac{\dot{u}}{\dot{y}} =\right) \frac{\alpha y}{x} = \frac{x}{\sigma}$.

10. Hence $\overline{\mathrm{AC}}^2 = tt = x \times \overline{x \pm s} = x \times \overline{t \pm r} = xq =$ DCT.

11. And $\overline{\mathrm{BC}}^2 = cc = y \times \overline{y + \sigma} = y \times \mathrm{C}t = d\mathrm{C}t$.

12. Conſequently $\frac{\overline{\mathrm{P}d}^2}{\mathrm{B}db} = \frac{xx}{\pm cc \mp yy} = \frac{tt}{cc} = \frac{\overline{\mathrm{AC}}^2}{\overline{\mathrm{BC}}^2}$.

13. Alſo $\overline{\mathrm{CE}}^2 = ww = \left(\overline{\mathrm{DT}}^2 \times \frac{\overline{\mathrm{QE}}^2}{\overline{\mathrm{PD}}^2} = \overline{\mathrm{DT}}^2 \times \frac{\mathrm{AE}a}{\mathrm{AD}a} =\right) ss \times \frac{tt - ww}{sx} = sx = \frac{tt}{cc} yy = \mathrm{AD}a = \mathrm{CDT}$.

14. Therefore $\mathrm{AE}a = \left(\frac{\overline{\mathrm{CE}}^2}{\overline{\mathrm{DT}}^2} \times \mathrm{AD}a = \frac{sx}{ss} \times sx =\right) xx = \overline{\mathrm{CD}}^2$.

15. And $\overline{\mathrm{QE}}^2 = \left(\frac{\overline{\mathrm{PD}}^2}{\mathrm{AD}a} \times \mathrm{AE}a =\right) \frac{cc}{tt} xx = \frac{p}{t} xx$.

In

In the general ſchemes. PLATE XV. and Fig. 5. PLATE XIV.

16. Let $\mathrm{A}a$, $\mathrm{B}b$, be the longeſt and ſhorteſt axes.
Draw $\mathrm{C}\varrho$ perpendicular to the tangent PT, cutting it in ϱ.
Put $\mathrm{CP}=\mathrm{T}$; $\mathrm{CQ}=\mathrm{C}$; $\mathrm{C}\varrho=\varrho$.

Then $\mathrm{C}\varrho=\varrho=\left(\frac{\mathrm{C}t\times\mathrm{CE}}{\mathrm{CQ}}=\frac{cc}{yy}\times\frac{ty}{c\mathrm{C}}=\right)\frac{tc}{\mathrm{C}}=\frac{\mathrm{AC}\times\mathrm{CB}}{\mathrm{CQ}}$.

Hence the parallelogram, under the two axes, is equal to the parallelogram under any two diameters.

17. Draw the tangents AN, an, to any vertices A, a, meeting any diameters $\mathrm{P}p$, $\mathrm{Q}q$, produced in V, U, and v, u, and the tangent PT in N, n.

Then $\mathrm{AU}=\left(\frac{\mathrm{CA}\times\mathrm{EQ}}{\mathrm{CE}}=\right)\frac{px}{p}$. And $\mathrm{AV}=\left(\frac{\mathrm{CA}\times\mathrm{PD}}{\mathrm{CD}}=\right)\frac{ty}{x}$.

18. Alſo $\mathrm{CU}=\left(\frac{\mathrm{AU}\times\mathrm{CQ}}{\mathrm{EQ}}=\right)\frac{c\mathrm{C}}{y}$. And $\mathrm{CV}=\left(\frac{\mathrm{CP}\times\mathrm{CA}}{\mathrm{CD}}=\right)\frac{t\mathrm{T}}{x}$.

19. Hence $\mathrm{PV}=(\mathrm{CV}\backsim\mathrm{CP}=)\,\mathrm{T}\times\frac{t\backsim x}{x}$. And $p\mathrm{V}=\mathrm{T}\times\frac{t+x}{x}$.

Alſo $\mathrm{QU}=(\mathrm{CU}\backsim\mathrm{CQ}=)\,\mathrm{C}\times\frac{c\backsim y}{y}$. And $q\mathrm{U}=\mathrm{C}\times\frac{c+y}{y}$.

20. When $\mathrm{A}a$ and $\mathrm{B}b$ are the longeſt and ſhorteſt axes; and when $y=p$,
Then $xx=tt\mp\frac{t}{p}yy$ will become $tt\mp pt=tt\mp cc$, which call ff.
And $\mathrm{CD}=x$, will become $\mathrm{CF}=\mathrm{C}f=f$.
The points F, f, are called the Focii.

21. Hence $\mathrm{AF}=af=\pm t\mp f$; $\mathrm{A}f=a\mathrm{F}=t+f$.

22. Alſo $\overline{\mathrm{CF}}^2=\overline{\mathrm{C}f}^2=ff=\pm tt\mp cc=\pm tt\mp pt$.
And in the ellipſis, $\overline{\mathrm{AC}}^2=\left(\overline{\mathrm{BC}}^2+\overline{\mathrm{CF}}^2=\right)\overline{\mathrm{BF}}^2$.
in the hyperbola, $\overline{\mathrm{CF}}^2=\left(\overline{\mathrm{AC}}^2+\overline{\mathrm{BC}}^2=\right)\overline{\mathrm{BA}}^2$.
Hence, a circle deſcribed from B, with the diſtance AC in the ellipſis, or from C, with the diſtance AB in the hyperbola, will cut the axis $\mathrm{A}a$ in the focii F, f.

23. Draw

23. Draw FP, $f\mathrm{P}$, from the focii F, f, to any point P of the curve; and draw the conjugate diameters $\mathrm{P}p$, $\mathrm{Q}q$.

Put $\mathrm{PF} = z$; $\mathrm{P}f = v$; $\mathrm{PC} = \mathrm{T}$; $\mathrm{QC} = \mathrm{C}$; $\frac{f}{t} = \varphi$; $\frac{f}{c} = \gamma$. Then

$$\overline{\mathrm{P}f}^2 = vv = yy + xx + 2xf + ff = tt + 2fx + \frac{ff}{tt}xx = \mathrm{TT} + ff + 2fx.$$

$$\overline{\mathrm{PF}}^2 = zz = yy + xx - 2xf + ff = tt - 2fx + \frac{ff}{tt}xx = \mathrm{TT} + ff - 2fx.$$

$$\text{For } yy = \left(\frac{cc}{tt} \times \overline{tt \backsim xx} = \right) \frac{tt \backsim ff}{tt} \times \overline{tt \backsim xx}.$$

24. $$\mathrm{P}f = v = \left(\sqrt{tt + 2fx + \frac{ff}{tt}xx} = \right) t + \frac{f}{t}x = t + \varphi x = \frac{tt + fx}{t}.$$

$$\mathrm{PF} = z = \left(\sqrt{tt - 2fx + \frac{ff}{tt}xx} = \right) t - \frac{f}{t}x = t - \varphi x = \frac{tt - fx}{t}.$$

25. $\mathrm{PF} \underset{\sim}{+} \mathrm{P}f = z \underset{\sim}{+} v = 2t = \mathrm{A}a.$

26. $\overline{\mathrm{P}f}^2 + \overline{\mathrm{PF}}^2 = vv + zz = 2yy + 2xx + 2ff = 2tt + 2\varphi\varphi xx = 2\mathrm{TT} + 2ff.$

27. $\overline{\mathrm{P}f}^2 - \overline{\mathrm{PF}}^2 = vv - zz = 4fx = \overline{v + z} \times \overline{v - z} = 2t \times \overline{v \overset{\sim}{+} z}.$

28. $$\mathrm{FP}f = zv = z \times \overline{2t \mp z} = 2tz \mp zz = tt \backsim \varphi\varphi xx = \overline{tt \backsim xx} \times \frac{cc}{tt}xx$$

$$= cc \backsim \frac{ff}{cc}yy = \frac{tt}{cc}yy + \frac{cc}{tt}xx = \overline{\mathrm{CE}}^2 + \overline{\mathrm{EQ}}^2$$

$$= \overline{\mathrm{CQ}}^2 = \mathrm{CC} = \mathrm{T} \times \mathrm{P}.$$

29. Let $$m = t - z = v - t = \frac{fx}{t} = \varphi x = \pm \tfrac{1}{2}v \mp \tfrac{1}{2}z$$

$$= \frac{fx}{\sqrt{cc + ff}} = \frac{x}{t}\sqrt{tt \mp cc} = \sqrt{tt \mp zv}$$

$$= \sqrt{tt \mp \mathrm{CC}} = \sqrt{ff + cc - zv}.$$

30. Hence $\dot{z} = -\dot{v} = \varphi\dot{x} = -\frac{f}{t}\dot{x} \pm \frac{fy}{px}\dot{y}.$

And $\dot{v} = -\dot{z} = \varphi\dot{x} = \frac{f}{t}\dot{x} = -\frac{fy}{px}\dot{y}.$

31. $\mathrm{AC} = t = \frac{1}{2}z \pm \frac{1}{2}v = \frac{cc}{p} = \sqrt{cc \pm ff} = \frac{fx}{m} = \frac{cx}{\sqrt{cc \backsim yy}}$

$$= \frac{c}{y}\sqrt{sx} = \sqrt{x \times \overline{x \backsim s}} = \frac{y^2 + \sqrt{y^4 + 4p^2x^2}}{2p} = \frac{yy}{2p} + \sqrt{\frac{y^4}{4p^2} + xx}.$$

32. $\overline{\mathrm{CD}}^2 = \overline{\mathrm{P}d}^2 = xx = \overline{\frac{ttyy}{ccs}}\Big|^2 = \overline{\frac{tyy}{ps}}\Big|^2 = \pm tt \mp \frac{t}{p}yy = \frac{tt}{cc} \times \overline{cc \mp yy}$

$$= \frac{ff + cc}{cc} \times \overline{cc \mp yy} = tt \mp \frac{tt}{cc}yy = \frac{mm}{\varphi\varphi} = \frac{tt}{ff}mm$$

$$= \frac{cc + ff}{ff}mm = \frac{tt}{ff} \times \overline{tt \mp zv} = \frac{tt}{ff} \times \overline{tt \mp \mathrm{CC}} = \overline{\frac{tt}{q}}\Big|^2$$

$$= \frac{tt}{cc} \times \overline{zv - \frac{tt}{cc}yy} = \frac{t}{p} \times \overline{zv - \frac{t}{p}yy} = \frac{tt}{ff} \times \overline{t - z}\Big|^2$$

$$= \frac{tt}{ff} \times \overline{v - t}\Big|^2 = \overline{\frac{tt \mp tz}{f}}\Big|^2 = \overline{t \mp u}\Big|^2 = \overline{\sqrt{tt + \frac{1}{4}ss} - \frac{1}{2}s}\Big|^2$$

$$= \frac{tt}{ff} \times \overline{tt \mp \frac{tt}{cc}\pi\pi} = \frac{t^4}{c^4} \times \frac{\mathrm{SS}}{\mathrm{RR}}\pi\pi.$$

33. And $\dot{x} = -\frac{ty\dot{y}}{px} = -\frac{tty\dot{y}}{ccx} = -\frac{tt\dot{y}}{qq} = -\frac{t\dot{z}}{f} = \frac{t\dot{v}}{f}.$

34. $\overline{\mathrm{PD}}^2 = \overline{\mathrm{C}d}^2 = yy = \frac{p}{t}sx = \frac{cc}{tt}sx = \frac{cc}{tt} \times \overline{\pm tt \mp xx} = \pm cc \mp \frac{cc}{tt}xx$

$$= \pm pt \mp \frac{p}{t}xx = \frac{2cc}{t}u \mp \frac{cc}{tt}uu = \frac{\pm tt \mp ff}{tt} \times \overline{\pm tt \mp xx}$$

$$= \frac{cc}{tt} \times \overline{zv - \frac{cc}{tt}xx} = \frac{cc}{tt} \times \overline{\mathrm{CC} - \frac{cc}{tt}xx} = \frac{cc}{ff} \times \overline{zv - cc}$$

$= \frac{cc}{ff}nn = \frac{pt}{ff}nn$ (putting $nn = zv - cc$)

$$= \frac{tt\pi\pi - c^4}{ff} = cc \mp \frac{tt}{cc} \times \frac{\mathrm{SS}}{\mathrm{RR}}\pi\pi = \frac{\mathrm{S'S'}}{\mathrm{RR}}\pi\pi.$$

35. And $\dot{y} = -\frac{px\dot{x}}{ty} = -\frac{ccx\dot{x}}{tty} = \frac{px\dot{z}}{fy} = -\frac{px\dot{v}}{fy}.$

36. Alſo $\frac{ff}{cc}yy = zv - cc = \frac{ff}{tt} \times \overline{tt \mp xx} = ff \mp \frac{ff}{tt}xx$

$$= nn = ff \mp mm = \mathrm{CC} - cc.$$

37. $\overline{\mathrm{PC}}^2 = \mathrm{TT} = xx + yy = xx \pm cc \mp \frac{cc}{tt}xx = tt - ff + \frac{ff}{tt}xx = cc + \varphi\varphi xx$
$= cc + tt \backsim zv = cc + mm = cc + tt \backsim \mathrm{CC}$
$= tp + \varphi\varphi xx = 2cc + ff \backsim zv = 2pt + ff \backsim zv.$

38. And $\dot{\mathrm{T}} = \frac{\varphi\varphi x\dot{x}}{\mathrm{T}} = \frac{\varphi\varphi x\dot{x}}{\sqrt{tp + \varphi\varphi xx}} = \left(\frac{\varphi\varphi x}{\mathrm{T}} \times -\frac{ty\dot{y}}{px} = \right)\frac{ffy\dot{y}}{pt\mathrm{T}} = -\frac{ffy\dot{y}}{cc\mathrm{T}}.$

39 Alſo $\overline{\mathrm{CQ}}^2 + \overline{\mathrm{CP}}^2 = (cc + tt =)\ \overline{\mathrm{CB}}^2 + \overline{\mathrm{CA}}^2.$

40. $\mathrm{CT} = q = \frac{tt}{x} = \left(tt \times \frac{f}{tm} = \right)\frac{ft}{m} = \frac{ft}{t \backsim z} = \frac{ft}{\sqrt{tt \backsim zv}}$

$\mathrm{C}t = \frac{cc}{y} = \frac{cf}{n} = \frac{cf}{\sqrt{zv - cc}}.$

41. And $\dot{q} = -\frac{q\dot{x}}{x} = -\frac{qq\dot{x}}{tt} = \frac{qqy\dot{y}}{ccx} = \frac{qqy\dot{y}}{ptx} = -\frac{tt\dot{x}}{xx}$ (for $tt = qx$).

42. $\mathrm{AT} = (\pm\mathrm{CT} \mp \mathrm{CA} =) \pm \frac{tt}{x} \mp t = \frac{t}{x} \times \pm\overline{t \mp x} = \pm\frac{ft}{m} \mp t = \pm\frac{ft}{t \mp z} \mp t.$

$a\mathrm{T} = (\mathrm{CT} + \mathrm{C}a =)\ \frac{tt}{x} + t = \frac{t}{x} \times \overline{t + x} = \frac{ft}{m} + t = \frac{ft}{t \backsim z} + t.$

43. $\mathrm{FT} = \left(\pm\mathrm{CT} \mp \mathrm{CF} = \pm\frac{ft}{m} \mp f = \frac{\pm ft \mp fm}{m}\right) = \frac{fz}{t \backsim z} = \frac{fz}{m} = \frac{tz}{x}.$

$f\mathrm{T} = \left(\mathrm{CT} + \mathrm{C}f = \frac{ft}{m} + f = \frac{ft + fm}{m} = \right)\frac{fv}{v \backsim t} = \frac{fv}{m} = \frac{tv}{x}.$

On Aa deſcribe a circle, draw the tangent TP′, and draw CP′. Continue DP to P′, and at right angles to TP′, draw FR′,

Then $\mathrm{FR}' = \left(\frac{\mathrm{FT}}{\mathrm{CT}} \times \mathrm{CP}' = \frac{tz}{x} \times t \times \frac{x}{tt} = \right) z = \mathrm{FP}.$

$fr' = \left(\frac{f\mathrm{T}}{\mathrm{CT}} \times \mathrm{CP}' = \frac{tv}{x} \times t \times \frac{x}{tt} = \right) v = f\mathrm{P}.$

44. $\mathrm{DT} = s = \frac{tyy}{px} = \frac{ttyy}{ccx} = \frac{\pm tt \mp xx}{x} = \frac{tnn}{fm}.$

$dt = \sigma = \frac{pxx}{ty} = \frac{ccxx}{tty} = \frac{cc \mp yy}{y} = \frac{cmm}{fn}.$

45. $\mathrm{AN} = \left(\frac{\mathrm{C}t}{\mathrm{CT}} \times \mathrm{AT} =\right) \frac{cm}{nx} \times \overline{t - x} = \frac{c}{n} \times \overline{f - m}.$

$an = \left(\frac{\mathrm{C}t}{\mathrm{CT}} \times a\mathrm{T} =\right) \frac{cm}{nx} \times \overline{t + x}.$

46. $\mathrm{VN} = \left(\frac{\mathrm{PV} \times \mathrm{C}t}{\mathrm{PC}} = \frac{t - x}{\mathrm{T}} \times \frac{\mathrm{T}}{x} \times \frac{fc}{n} =\right) \frac{fc}{nx} \times \overline{t - x} = \frac{fc}{nm} \times \overline{f - m}$

$= \frac{fc}{\sqrt{zv - cc} \times \sqrt{tt - zv}} \times \overline{f - \sqrt{tt - zv}}$

$\mathrm{UN} = \mathrm{C}t = \frac{cc}{y}.$

47. $\mathrm{AD} = (\pm \mathrm{AC} \mp \mathrm{CD} = \pm t \mp \frac{tm}{f} =) \frac{t}{f} \times \overline{\pm f \mp m}$

$a\mathrm{D} = (\mathrm{AC} + \mathrm{CD} = t + \frac{tm}{f} =) \frac{t}{f} \times \overline{f + m}.$

48. Produce PF, pf, ſo meet the curve in Π, π;

Draw $\Pi\Delta$ perpendicular to Aa.

Put $\mathrm{F}\Pi = z'$, $f\pi = v'$, $\mathrm{F}\Delta = x'$.

Now, $\mathrm{FD} = (\pm \mathrm{CF} \mp \mathrm{CD} =) \pm f \mp x = \pm f \mp \frac{tm}{f} = \frac{tz - cc}{f}$

$f\mathrm{D} = f + \frac{tm}{f} = \frac{tv - cc}{f}.$

Then $\mathrm{F}\Delta = x' = (\mathrm{C}\Delta - \mathrm{CF} =) \frac{tm}{f} - f = \left(\frac{t}{f} \times \overline{t - z'} - f =\right) \frac{cc - tz}{f}$

$f\Delta = f + \frac{tm}{f} = \frac{tv' - cc}{f}.$

49. From F, f, draw FR, fr, perpendicular to the tangent PT, and cutting it in R, r.

$$\mathrm{FR} = \lambda = \frac{\mathrm{FT}\times \mathrm{C}\varrho}{\mathrm{CT}} = \frac{cz}{\mathrm{C}} = \frac{cz}{\sqrt{zv}} = \frac{cz}{\sqrt{2tz - zz}} = c\sqrt{\frac{z}{v}}$$

$$= c\sqrt{\frac{t-\varphi x}{t+\varphi x}} = \sqrt{\frac{ptz}{zt - z}} = \frac{z\varrho}{t}$$

$$fr = \lambda' = \frac{f\mathrm{T}\times \mathrm{C}\varrho}{\mathrm{CT}} = \frac{cv}{\mathrm{C}} = \frac{cv}{\sqrt{zv}} \cdots\cdots = c\sqrt{\frac{z}{v}}$$

$$= e\sqrt{\frac{t+\varphi x}{t-\varphi x}} = \sqrt{\frac{ptv}{2t - v}}.$$

$\dot{\lambda} = \frac{tc\dot{z}}{v\sqrt{zv}}$. For $\lambda^2 = \frac{c^2 z}{v}$. Th. $2\lambda\dot{\lambda} = \left(\frac{c^2 v\dot{z} - c^2 z\dot{v}}{vv} =\right) \frac{2c^2 t\dot{z}}{vv}$.

50. $\mathrm{TR} = \left(\frac{\mathrm{AT}\times \mathrm{RF}}{\mathrm{AN}} = \frac{tnz}{m\mathrm{C}} =\right) \frac{ttnz}{f\mathrm{C}x} = \frac{ttn}{fx}\sqrt{\frac{z}{v}} = \frac{tn}{m}\sqrt{\frac{z}{v}}$.

$$\mathrm{T}r = \frac{a\mathrm{T}\times rf}{an} = \frac{tn}{m}\sqrt{\frac{v}{z}}.$$

51. Draw PM perpendicular to the tangent PT, meeting the axes Aa, Eb, in M, m.

$$\mathrm{DM} = \left(\frac{\mathrm{PD}\times \mathrm{PD}}{\mathrm{TD}} =\right) \frac{ccm}{ft} = \frac{cc}{ft}\sqrt{tt \pm zv} = \frac{cc}{tt}x = \frac{p}{t}x$$

$$= \frac{\mathrm{S}}{\mathrm{R}}\pi = p \mp \frac{p}{t}u = \frac{cc}{tf}\sqrt{tt \mp \frac{tt}{cc}\pi\pi}.$$

$$dm = \left(\frac{\mathrm{PD}\times \mathrm{P}d}{\mathrm{DM}} =\right) \frac{ttn}{fc} = \frac{tt}{cc}\sqrt{zv - cc} = \frac{tt}{cc}y = \frac{t}{p}y.$$

52. $\mathrm{CM} = \left(\mathrm{CD} \mp \mathrm{DM} = x \mp \frac{cc}{tt}x =\right) \frac{tt \mp cc}{tt}x = \frac{ff}{tt}x = \frac{f}{t}m$

$$= \frac{f}{t}\sqrt{tt \mp zv} = \frac{f}{t}\times \overline{t \mp z} = \frac{f}{t}\times\sqrt{tt \mp \frac{tt}{cc}\pi\pi}$$

$$\mathrm{C}m = \left(dm \mp \mathrm{C}d = \frac{tt}{cc}y \mp y =\right) \frac{tt \mp cc}{cc}y = \frac{ff}{cc}y = \frac{f}{c}n = \frac{f}{c}\sqrt{zv - cc}$$

53. $\mathrm{FM} = \left(\mathrm{CF} \backsim \mathrm{CM} = f \backsim \frac{fm}{t} =\right) \frac{f}{t}\times\overline{t \mp m} = \frac{fz}{t} = \pm f \mp \frac{ff}{tt}x$.

$$f\mathrm{M} = \left(f\mathrm{C} + \mathrm{CM} = f + \frac{fm}{t} =\right) \frac{f}{t}\times\overline{t + m} = \frac{fv}{t} = f \pm \frac{ff}{tt}x.$$

54. $\overline{\mathrm{F}m}^2 = \overline{fm}^2 = (\overline{\mathrm{C}f}^2 + \overline{\mathrm{C}m}^2 =)\ ff + \frac{ffnn}{cc} = \frac{ffzv}{cc} = \frac{ff}{cc} \times \mathrm{CC}.$

55. $\mathrm{AM} = (\mathrm{AC} \backsim \mathrm{CM} =)\ t \backsim \frac{fm}{t} = \frac{tt \backsim fm}{t} = t \backsim \frac{ff}{tt}x.$

$a\mathrm{M} = (a\mathrm{C} + \mathrm{CM} =)\ t + \frac{fm}{t} = \frac{tt + fm}{t} = t + \frac{ff}{tt}x.$

56. $\mathrm{TM} = (\mathrm{TF} + \mathrm{FM} = \frac{fz}{m} + \frac{fz}{t} =)\ \frac{fzv}{tm} = \frac{zv}{x} = \frac{\mathrm{CC}}{x}$

$tm = (\mathrm{C}t + \mathrm{C}m = \frac{fc}{n} + \frac{fn}{c} =)\ \frac{fzv}{cn} = \frac{zv}{y} = \frac{\mathrm{CC}}{y}.$

57. $\mathrm{PM} = \pi = \left(\frac{\mathrm{FR} \times \mathrm{TM}}{\mathrm{FT}} =\right)\ \frac{c}{t}\sqrt{zv} = \frac{c}{t}\mathrm{C} = \frac{p}{c}\mathrm{C} = \frac{c}{t}\sqrt{tt \backsim \varphi^2 x^2}$

$= \frac{c}{t}\sqrt{\overline{t + \varphi x} \times \overline{t - \varphi x}} = \frac{cc}{tt} \times \frac{\mathrm{R}}{\mathrm{S}}x = \frac{c\mathrm{R}}{t\mathrm{S}}\sqrt{ce \mp yy}$

$= \frac{c}{tt}\sqrt{t^4 - ffxx} = \frac{c}{t}\sqrt{cc + \gamma\gamma yy} = \frac{1}{t}\sqrt{c^4 + ffyy}$

$= \frac{c}{t}\sqrt{2tz \mp zz} = \sqrt{2pz \mp \frac{p}{t}zz}.$

$\mathrm{P}m = \pi' = \left(\frac{\mathrm{PM} \times \mathrm{P}d}{\mathrm{DM}} =\right)\ \frac{t}{c}\sqrt{zv} = \frac{t}{c}\mathrm{C} = \frac{c}{p}\mathrm{C} = \frac{t}{c}\sqrt{tt \backsim \varphi\varphi xx}$

$= \frac{t}{c}\sqrt{\overline{t + \varphi x} \times \overline{t - qx}} = \frac{1}{c}\sqrt{t^4 \backsim ffxx} = \frac{t}{c}\sqrt{cc + \gamma\gamma yy}$

$= \frac{t}{cc}\sqrt{c^4 + ffyy} = \frac{t}{c}\sqrt{2tz \mp zz} = \sqrt{\frac{2tt}{p}z \mp \frac{t}{p}zz}.$

58. $\mathrm{M}m = (\mathrm{P}m \mp \mathrm{PM} = \frac{t}{c}\mathrm{C} \mp \frac{c}{t}\mathrm{C} =)\ \frac{tt \mp cc}{ct}\mathrm{C} = \frac{ff}{ct}\sqrt{zv}.$

$= \frac{ff}{ct}\sqrt{tt \backsim \varphi\varphi xx}.$

59. $\mathrm{PT} = \tau = \left(\frac{\mathrm{PD} \times \mathrm{PM}}{\mathrm{DM}} =\right)\ \frac{n}{m}\mathrm{C} = \sqrt{\frac{zv - cc}{tt - zv} \times zv} = \frac{tn}{fx}\mathrm{C}.$

$= \frac{ty}{cx}\mathrm{C} = \frac{ty}{cx}\sqrt{tt \backsim \varphi\varphi xx} = \frac{\sqrt{\overline{zv - cc} \times zv}}{t \mp x}.$

$\mathrm{P}t = \tau' = \left(\frac{\mathrm{PT} \times \mathrm{P}d}{\mathrm{DT}} =\right)\ \frac{m}{n}\mathrm{C} = \sqrt{\frac{tt - zv}{zv - cc} \times vz} = \frac{fx}{tn}\mathrm{C}$

$= \frac{cx}{ty}\mathrm{C} = \frac{cx}{ty}\sqrt{tt - \varphi\varphi xx}.$

60. $$\mathrm{TN} = \left(\frac{\mathrm{PT}\times\mathrm{TA}}{\mathrm{DT}} = \frac{n\mathrm{C}}{m}\times\frac{t}{m}\times\overline{f\backsim m}\times\frac{fm}{tnn} = \right)\frac{f\mathrm{C}}{mn}\times\overline{f\backsim m} = \frac{c\mathrm{C}}{xy}\times\overline{t\backsim x}$$

$$\mathrm{T}n = \left(\frac{\mathrm{PT}\times\mathrm{T}a}{\mathrm{DT}} = \frac{n\mathrm{C}}{m}\times\frac{t}{m}\times\overline{f+m}\times\frac{fm}{tnn} = \right)\frac{f\mathrm{C}}{mn}\times\overline{f+m} = \frac{c\mathrm{C}}{xy}\times\overline{t+x}$$

61. $$\mathrm{PN} = \left(\frac{\mathrm{AD}\times\mathrm{PT}}{\mathrm{DT}} = \frac{t}{f}\times\overline{f\backsim m}\times\frac{n\mathrm{C}}{m}\times\frac{fm}{tnn} = \right)\frac{\mathrm{C}}{n}\times\overline{f\backsim m} = \frac{c\mathrm{C}}{ty}\times\overline{t\backsim x}$$

$$\mathrm{P}n = \left(\frac{a\mathrm{D}\times\mathrm{PT}}{\mathrm{DT}} = \frac{t}{f}\times\overline{f+m}\times\frac{n\mathrm{C}}{m}\times\frac{fm}{tnn} = \right)\frac{\mathrm{C}}{n}\times\overline{f+m} = \frac{c\mathrm{C}}{ty}\times\overline{t+x}$$

62. $$\mathrm{PR} = \left(\frac{\mathrm{PT}\times\mathrm{FM}}{\mathrm{TM}} = \right)\frac{nz}{\mathrm{C}} = n\sqrt{\frac{z}{v}} = \sqrt{\frac{z}{v}\times\overline{zv-cc}} = \frac{f}{c}y\times\sqrt{\frac{z}{v}}$$

$$= \frac{f}{c}y\times\sqrt{\frac{t-\varphi x}{t+\varphi x}} = \frac{f}{cc}\times\lambda y$$

$$\mathrm{P}r = \left(\frac{\mathrm{PT}\times f\mathrm{M}}{\mathrm{TM}} = \right)\frac{nv}{\mathrm{C}} = n\sqrt{\frac{v}{z}} = \sqrt{\frac{v}{z}\times\overline{zv-cc}} = \frac{f}{c}\times y\sqrt{\frac{v}{z}} = \frac{f}{c}y\times\sqrt{\frac{t+\varphi x}{t-\varphi x}}$$

63. $$\mathrm{T}\varrho = \left(\frac{\mathrm{PT}\times\mathrm{TC}}{\mathrm{TM}} = \right)\frac{ntt}{m\mathrm{C}} = \frac{tt}{t-z}\sqrt{\frac{zv-ce}{vz}} = \frac{fytt}{mc\mathrm{C}} = \frac{yt^3}{xc\mathrm{C}}$$

64. $$\mathrm{P}\varrho = \left(\mathrm{T}\varrho \mp \mathrm{TP} = \frac{ntt}{m\mathrm{C}} \mp \frac{n\mathrm{C}}{m}\times\right)\frac{n}{m\mathrm{C}}\times\overline{tt \mp \mathrm{CC}} = \frac{n}{m\mathrm{C}}\times mm = \frac{nm}{\mathrm{C}}.$$

65. $$\mathrm{T}t = \left(\frac{\mathrm{PT}\times\mathrm{C}t}{\mathrm{PD}} = \frac{n\mathrm{C}}{m}\times\frac{cf}{n}\times\frac{f}{cn} = \right)\frac{ff\mathrm{C}}{mn} = ff\sqrt{\frac{zv}{\overline{tt-zv}\times\overline{zv-cc}}}$$

$$= \frac{tc}{xy}\mathrm{C} = \frac{tc}{xy}\sqrt{tt-\frac{ff}{tt}xx}$$

66. $$\mathrm{N}n = \left(\frac{\mathrm{PT}\times\mathrm{A}a}{\mathrm{DT}} = \right)\frac{2f\mathrm{C}}{cn} = \frac{2c}{y}\mathrm{C} = \frac{2c}{y}\sqrt{zv} = 2f\sqrt{\frac{zv}{zv-cc}}$$

From C and F, draw Cγ and Fε, perpendicular to Pf
Put Pε = b; Σ = *co*-ſine of the ∠ AfP; R = tabular radius..

Then $$f\gamma = \left(\frac{\mathrm{C}f\times f\mathrm{D}}{\mathrm{P}f} = \frac{t\,\overline{v-cc}}{f} = \right)\frac{tv-cc}{u} = \left(\frac{\Sigma}{\mathrm{R}}\times\mathrm{C}f\right)\frac{\Sigma f}{\mathrm{R}}$$

And $$\mathrm{P}f = v = \frac{cc}{t-\frac{\Sigma}{\mathrm{R}}f} = \frac{tt-ff}{t-\frac{\Sigma}{\mathrm{R}f}}$$

P

$$\text{P}s = h = \left(\frac{\pm zz + vv \mp 4ff}{2v} = \frac{\pm \text{TT} \mp ff}{v} = \frac{\pm 2pt \mp vz}{v} = \right)$$

$$= \pm \frac{2pt}{v} \mp z = \pm \frac{z+v}{v} p \mp z.$$

Hence $p = \frac{\pm h \mp z}{z+v} \times v$; $v = \frac{pz}{\mp h \pm z - p}$; $z = \frac{p-h}{v-p} \times v$

Therefore p is leſs than, equal to, or greater than $\pm h \mp z$, in the Ellipſis, Parabola, or Hyperbola.

67. Draw $m\mu$, making the $\angle \text{P}m\mu = \angle \text{PCM}$, and cutting PC in μ.

Then $\text{P}\mu = \left(\frac{\text{PM} \times \text{P}m}{\text{PC}} = \right) \frac{\text{CC}}{\text{T}} = \frac{zv}{\sqrt{tt \pm cc \mp zv}} = \frac{tt - \varphi\varphi xx}{\sqrt{cc + \varphi\varphi xx}}$

$= \text{P} = \frac{1}{2}$ parameter to PC.

68. Let FΦ be an ordinate to the axis Aa. at the focus F. and ΦG a tangent to the curve in Φ, meeting Aa, BC, AN, an, in G, g, S, s. Then

$$\text{F}\Phi = \sqrt{\frac{cc}{tt} \times \text{AF}a} = \sqrt{\frac{cc}{tt} \times cc} = \Big) \frac{cc}{t} = \frac{pt}{t} = p$$

$= \frac{1}{2}$ parameter to Aa.

69. $\text{CG} = \frac{tt}{f} = \frac{ff \pm cc}{f} = \frac{tt}{\sqrt{tt - cc}} = \frac{tt}{\sqrt{t \times t - p}}$

70. $\text{FG} = (\text{CG} \backsim \text{CF} = \frac{tt}{f} \backsim f = \frac{tt \backsim ff}{f} = \frac{cc}{f} = \frac{pt}{f}.$

71. Draw PH parallel to Aa; and GH perpendicular to Aa, meeting, PH, PT, in H, h; then

$$\text{PH} = \text{DG} = (\text{CG} \backsim \text{DC} =) \frac{tt}{f} \backsim x = \frac{t}{f} z = \frac{t}{f} \times \text{PF}.$$

72. TG

72. $\mathrm{TG} = (\mathrm{CT} \backsim \mathrm{CG} = \frac{ft}{m} \backsim \frac{tt}{f} = \frac{t}{m} \times \overline{f \backsim x} =) \frac{t}{t-z} \times \frac{\pm tz \mp cc}{f} = \frac{tt}{fx} \times \overline{f \backsim x}.$

73. $\mathrm{G}b = (\frac{\mathrm{PD} \times \mathrm{TG}}{\mathrm{DT}} = \frac{cn}{f} \times \frac{t}{m} \times \overline{f \backsim x} \times \frac{fm}{tnn} = \frac{\mathrm{C}}{n} \times \overline{f \backsim x} =)$

$$\frac{cc}{fy} \times \overline{f \backsim x} = \frac{c}{\sqrt{zv-cc}} \times \frac{tz-cc.}{f}$$

74. $\mathrm{P}b = (\frac{\mathrm{TP} \times \mathrm{DG}}{\mathrm{DT}} =) \frac{z}{n}\mathrm{C} = \frac{cz}{fy}\mathrm{C} = z \times \sqrt{\frac{zv}{zv-cc}} = \frac{cz}{fy} \sqrt{zv}.$

75. $\mathrm{R}b = (\mathrm{P}b - \mathrm{PR} =) \frac{z\mathrm{C}}{n} - \frac{zn}{\mathrm{C}} = \frac{z}{n\mathrm{C}} \times \overline{\mathrm{CC}-nn} =)$

$$\frac{zcc}{n\mathrm{C}} = \frac{zcc}{\sqrt{\overline{zv-cc}+zv}} = \frac{zc^3}{fy\mathrm{C}}$$

76. $\mathrm{F}b = (\sqrt{\overline{\mathrm{FR}}^2 + \overline{\mathrm{R}b}^2} =) \frac{cz}{n} = \frac{cz}{\sqrt{zv-cc}} = \frac{ccz}{fy} = \frac{ptz.}{fy}$

77. Let $\mathrm{D}\Sigma$ be any Ordinate to the axe $\mathrm{A}a$, cutting the curve in Σ, and the focal tangent $\Phi\mathrm{G}$ in σ;

Then $\mathrm{D}\sigma = (\frac{\mathrm{F}\Phi + \mathrm{DG}}{\mathrm{FG}} = \frac{cc}{t} \times \frac{tz}{f} \times \frac{f}{cc} =) z = \mathrm{FP} = \mathrm{F}\Sigma.$

78. Therefore $\mathrm{AS} = \mathrm{AF}$; $as = a\mathrm{F}$; $\mathrm{C}g = \mathrm{CA}$; by ſim. △s.

79. $\overline{\mathrm{DF}}^2 = (\overline{\mathrm{D}\sigma}^2 - \overline{\mathrm{D}\Sigma}^2 = \overline{\mathrm{D}\sigma + \mathrm{D}\Sigma} \times \overline{\mathrm{D}\sigma - \mathrm{D}\Sigma} =) \mathrm{P}\sigma \times \sigma\Sigma.$

79. Let the tangents PN, $p\mathrm{L}$, to the oppoſite vertices P, p, cut the tangents AN, an, to the oppoſite vertices A, a, in N, n, L, l.
Then $\mathrm{P}n = p\mathrm{L}$; $an = \mathrm{AL}$; $\mathrm{PN} = pl$; $\mathrm{AN}\, al$.
For the Trapezia's $\mathrm{PC}an$, $p\mathrm{CAL}$, are ſimilar and equal;
And ſo are the Trapezia's PCAN, $p\mathrm{C}al$.

81. $\mathrm{PF}\times\mathrm{M}f=\left(\frac{f}{t}\times vz=\right)\mathrm{P}f\times\mathrm{MF}=\frac{f}{t}\times\mathrm{FP}f.$

82. $\mathrm{CEQ}=(xy=)\ \mathrm{CDP}=d\,\mathrm{CD}=\mathrm{DM}\times dm.$

83. $\mathrm{AN}\times a\mathrm{D}=\frac{ctn}{f}=an\times\mathrm{AD}=\mathrm{AC}\times\mathrm{PD}=\mathrm{D}al=\mathrm{DAL}.$

84. $\mathrm{RP}r=nn=\mathrm{TP}\varrho.$

85. $\mathrm{RT}r=\left(\frac{nn}{mm}\times tt=\frac{tt-xx}{xx}\times tt=\right)\mathrm{AT}a.$

86. $\mathrm{RT}f=\left(\frac{tnz}{m\mathrm{C}}\times\frac{fv}{m}=\frac{ft}{m}\times\frac{n\mathrm{C}}{m}=\right)\mathrm{CTP}=r\,\mathrm{TF}.$

87. $\mathrm{NT}n=\left(\frac{ff\mathrm{CC}}{mm\,nn}\times\overline{ff\backsim mm}=\right)\frac{ff\mathrm{CC}}{mm}=\mathrm{PT}f.$

88. $\mathrm{PN}\times\mathrm{CB}=\left(\overline{t\backsim x}=\frac{fc}{tn}\mathrm{C}=\right)\mathrm{AN}\times\mathrm{CQ}.$

89. $\mathrm{CAV}=\left(t\times\frac{ty}{x}=\frac{tt}{x}\times y=\right)\mathrm{CT}\times\mathrm{PD}.$

90. $\mathrm{PR}h=\left(\frac{cczz}{\mathrm{CC}}=\right)\overline{\mathrm{FR}}^2.$

Hence $\mathrm{F}h$ is perpendicular to FP.

91. $\mathrm{FM}f=\left(\frac{fz}{t}\times\frac{fv}{t}=\right)\frac{ff}{tt}\times zv=\frac{ff}{tt}\times\mathrm{CC}=\frac{ff}{tt}\times\overline{\mathrm{CQ}}^2$

92. $\mathrm{AM}a=\left(tt\backsim\frac{ffmm}{tt}=\right)cc\times\frac{ff}{tt}\times zv$

$$=\overline{\mathrm{CB}}^2+\frac{ff}{tt}\times\overline{\mathrm{CQ}}^2=\overline{\mathrm{CB}}^2+\mathrm{FM}f.$$

93. $\frac{\overline{PN}^2}{\overline{TN}^2} = \frac{xx}{tt} = \frac{\overline{CD}^2}{\overline{AC}^2} = \frac{\overline{AN}^2}{\overline{VN}^2} = \frac{\overline{CA}^2}{\overline{CT}^2} = \frac{ADa}{ATa} = \frac{ADa}{RTr} = \frac{\overline{AD}^2}{\overline{AT}^2} = \frac{\overline{aD}^2}{\overline{aT}^2} = \frac{CDT}{ATa}$.

94. $\frac{FT}{CT} = \frac{z}{t} = \frac{FP}{CA} = \frac{FR}{C\varrho}$. Sim. $\triangle$s TFR, TCϱ.

And $\frac{AD}{DT} = \frac{x}{t+x} = \frac{CD}{aD} = \frac{AC}{aT}$.

95. $\frac{fT}{CT} = \frac{v}{t} = \frac{fP}{AC} = \frac{fr}{C\varrho}$. And $\frac{AT}{DT} = \frac{t}{t+x} = \frac{AC}{aD} = \frac{CT}{aT}$.

96. $\frac{PM}{Pm} = \frac{cc}{tt} = \frac{p}{t} = \frac{\overline{BC}^2}{\overline{AC}^2}$. And $\frac{Pm}{fm} = \frac{t}{f} = \frac{AC}{CF}$.

97. $\frac{FMf}{\overline{PM}^2} = \frac{ff}{cc} = \frac{\overline{t+c} \times \overline{t-c}}{cc} = \left(\frac{tt-tp}{tp} =\right) \frac{t-p}{p} = \frac{\overline{CF}^2}{\overline{CB}^2}$.

98. Let PM, the perpendicular to the tangent PT, cut the axis Aa in M; and fP, produced, cut FR in φ.
Then will PM biſect the angle FPf.
For PF × Mf = Pf × MF.

99. And the angle FPT is equal to the angle fPt.
For ∠TPF + ∠FPM = ∠tPf + ∠fPM.

100. Therefore PT will biſect the angle FPφ.
For ∠φPT = (∠fPt =) ∠FPT.

101. Conſequently Pφ = PF. And Rφ = RF.

102. A circumference of a circle deſcribed from t, with the radius tN, will cut the axis Aa in the focii F, f.
For TN × Tn = TF × Tf.

103. A circumference of a circle deſcribed from C, with the radius CA, will cut the tangent PT in R, *r*.

Whence the perpendiculars RF, *rf*, to that tangent, will cut the axis A*a* in the focii F, *f*.

For $\mathrm{TR}\times\mathrm{T}r = \mathrm{TA}\times\mathrm{T}a$.

104. A circumference deſcribed from B, with AC, in the Ellipſis, or from C, with AB in the Hyperbola, will cut the axis A*a*, in the focii F, *f*.

For, in the Ellipſis, $tt = cc + ff$, or $\overline{\mathrm{AC}}^2 = (\overline{\mathrm{BC}}^2 + \overline{\mathrm{CF}}^2 =)\ \overline{\mathrm{BF}}^2$.

And in the Hyperbola, $ff = tt + cc$, or $\overline{\mathrm{CF}}^2 = (\overline{\mathrm{AC}}^2 + \overline{\mathrm{BC}}^2 =)\ \overline{\mathrm{AB}}^2$.

105. Let CQ produced, cut PF, P*f*, in *z*, *x*; draw *mz*, *mx*, and MZ, MX, parallel to them, cutting PF, P*f* in Z, X.

Then $\mathrm{P}x = \mathrm{CR} = \mathrm{CA} = \mathrm{P}z = t$.

106. Hence $\angle \mathrm{P}xz = \angle \mathrm{P}zx$; and $\angle mxz = \angle mzx$.

For P*m* is perpendicular to *zx*.

Conſequently, the angles P*zm*, P*xm*; PZM, PXM are equal.

107. And the triangles P*zm*, P*xm*, are ſimilar and equal:

And ſo are the triangles PZM, PXM.

Conſequently, the trapezias P*zmx*, PZMX, are ſimilar.

108. Let CR, C*r*, cut PF, P*f*, in K, *k*.

Then $\mathrm{CK} = \left(\frac{\mathrm{CF}\times\mathrm{P}f}{\mathrm{F}f} =\right) \frac{1}{2}\mathrm{P}f = \mathrm{P}k = kr$.

And $\mathrm{C}k = \left(\frac{\mathrm{C}f\times\mathrm{PF}}{\mathrm{F}f} =\right) \frac{1}{2}\mathrm{PF} = \mathrm{PK} = \mathrm{KR}$.

109. Alſo $\mathrm{PX} = \mathrm{PZ} = \left(\frac{\mathrm{PM}}{\mathrm{P}m}\times\mathrm{P}z = \frac{cC}{t}\times\frac{c}{tC}\times t =\right) \frac{cc}{t} = \left(\frac{pt}{t} =\right) p$.

110. The Trapezias $\mathrm{FG}b\mathrm{R}$, PDFR, $frt\mathrm{C}$, are ſimilar, and conſequently their correſponding parts are proportional.

That is, $\left\{\begin{array}{c} \frac{\mathrm{FG}}{\mathrm{PD}} = \frac{\mathrm{G}b}{\mathrm{DF}} = \frac{b\mathrm{R}}{\mathrm{FR}} = \frac{\mathrm{RF}}{\mathrm{RP}} \\ \overline{fr} = \overline{rt} = \overline{t\mathrm{C}} = \overline{\mathrm{C}f} \end{array}\right\}$

For the triangles $\mathrm{RF}b$, RPF, and $\mathrm{F}b\mathrm{G}$, FPD, are ſimilar.

111. The Trapezias $\mathrm{C}\varrho\mathrm{PD}$, $t\mathrm{CMP}$ are ſimilar, and conſequently their correſponding parts are proportional.

That is, $\frac{\mathrm{C}\varrho}{t\mathrm{C}} = \frac{\varrho\mathrm{P}}{\mathrm{CM}} = \frac{\mathrm{PD}}{\mathrm{PM}} = \frac{\mathrm{DC}}{\mathrm{P}t}$.

112. And CR, $\mathrm{C}r$, are parallel to $\mathrm{P}f$, PF, and equal to CA.

For $\mathrm{RT}f = \mathrm{CTP} = r\mathrm{TF}$.

113. Let $\Sigma' =$ ſine of the $\angle t\mathrm{P}f$, or TPF; $\mathrm{R} =$ tabular radius.

Then $\frac{\mathrm{R}}{\Sigma'} = \left(\frac{\mathrm{P}f}{fr} =\right) \frac{\mathrm{C}}{c} = \frac{\sqrt{zv}}{c}$.

Put $\Sigma =$ ſine of the $\angle \mathrm{PCQ}$, made by any diameter and its ordinate.

Then $\frac{\mathrm{TT}}{\mathrm{CC}} = \frac{1}{2}ff \underset{\sim}{\pm} \sqrt{\frac{1}{4}f^4 \backsim \frac{r^2t^2c^2}{\Sigma^2}}$; $\frac{tt}{cc} = \frac{1}{2}\mathrm{FF} \underset{\sim}{\pm} \sqrt{\frac{1}{4}\mathrm{F}^4 \backsim \frac{\mathrm{T}^2\mathrm{C}^2\Sigma^2}{rr}}$.

114. Let the parallels $\mathrm{PF}\Pi$, $\mathrm{C}\beta r$, $pf\pi$ be drawn, cutting the curve in Π, β, π; and ordinately applied to ſome diameter (2τ), whoſe parameter is 2π, and ſemi-conjugate $\mathrm{C}\beta = \chi$, to which $\mathrm{P}\pi$ is ordinately applied at δ.

Then $\mathrm{C}\delta = \left(\frac{\mathrm{PF}+\pi f}{2} = \frac{\mathrm{PF}+\mathrm{F}\Pi}{2} =\right) \frac{1}{2}z + \frac{1}{2}z' = \frac{1}{2}\mathrm{P}\Pi = \left(\frac{\overline{\mathrm{C}\beta}^2}{\mathrm{C}r} =\right) \frac{\overline{\mathrm{C}\beta}^2}{\mathrm{AC}} = \frac{\chi\chi}{t} = \frac{\pi\tau}{t}$.

115. $\overline{\mathrm{C}\beta}^2 = \left(\frac{1}{2}\mathrm{AC} \times \mathrm{P}\Pi =\right) \frac{1}{2}t \times \overline{z+z'} = \left(\overline{\mathrm{AC}}^2 \times \frac{\mathrm{PF}\Pi}{\mathrm{AF}a} =\right) \frac{tt}{cc}zz' = \frac{t}{p}zz'$.

116. $\frac{\mathrm{FP}}{\mathrm{F}\Pi} = \frac{z}{z'} = \frac{\mathrm{RD}}{\mathrm{F}\Delta} = \frac{tz-cc}{cc-tz'} = \frac{z-p}{p-z'}$.

117. $\mathrm{PF}\Pi = \frac{pzz}{2z-p} = \frac{1}{2}p \times \overline{z+z'} = \frac{1}{2}p \times \mathrm{P}\Pi$.

118. $\mathrm{P}\Pi = z + z' = \frac{2zz}{2z-p} = \frac{2\pi\tau}{t} = \frac{\tau}{t} \times 2\pi$.

$$\overline{\mathrm{CB}}^2 = cc = pt = \pm tt \mp ff = \pm\overline{t \mp f}\times\overline{t+f} = \mathrm{AF}a = \mathrm{V}\mathrm{AU} = t\mathrm{C}d = \mathrm{CFG}$$

$$= \mathrm{PM}\times\mathrm{C}\varrho = \mathrm{FR}\times fr = \mathrm{AN}\times an = al\times\mathrm{AL} = \mathrm{AC}\times\mathrm{F}\Phi = tt\times\frac{\mathrm{PM}}{\mathrm{P}m}$$

$$= tt\times\frac{\mathrm{DM}}{\mathrm{DC}} = tt\times\frac{d\mathrm{C}}{dm} = tt\times\frac{\overline{\mathrm{PM}}^2}{\overline{\mathrm{CQ}}^2} = tt\times\frac{\overline{\mathrm{PD}}^2}{\mathrm{AD}a} = tt\times\frac{\overline{\mathrm{PD}}^2}{\mathrm{TDC}} = ff\times\frac{\mathrm{PD}}{\mathrm{C}m}$$

$$= \overline{\frac{\mathrm{CQ}\times\mathrm{C}\varrho}{\mathrm{AC}}}^{\,2} = \overline{\frac{\mathrm{AN}\times\mathrm{CQ}}{\mathrm{PN}}}^{\,2} = \mathrm{AM}a - \mathrm{FM}f = \mathrm{FP}f - \mathrm{RP}r = ff\times\frac{t}{t-p}$$

$$= \mathrm{TT}+\mathrm{CC}-tt = \frac{tt}{xx}\times\overline{zv+xx-tt} = \tfrac{1}{2}zv \pm \sqrt{\tfrac{1}{4}z^2v^2ffyy}.$$

$$\overline{\mathrm{CQ}}^2 = \mathrm{CC} = \mathrm{P}\times\mathrm{T} = \mathrm{MP}m = \mathrm{NP}n = \mathrm{TP}t = \mathrm{FP}f = \mathrm{CP}\eta = \mathrm{L}pl$$

$$= \mathrm{PN}\times p\mathrm{L} = \mathrm{CD}\times\mathrm{TM} = \mathrm{PD}\times tm = \frac{tt}{cc}\times\overline{\mathrm{PM}}^2 = \frac{cc}{tt}\times\overline{\mathrm{P}m}^2 = \frac{cc}{ff}\times\overline{fm}^2$$

$$= \frac{tt}{ff}\times\mathrm{FM}f = \overline{\frac{ct}{ff}\times\mathrm{M}m}^{\,2} = \overline{\frac{\mathrm{ACB}}{\mathrm{C}\varrho}}^{\,2} = \overline{\mathrm{PM}}^2+\mathrm{FM}f = \overline{\mathrm{BC}}^2+\mathrm{RP}r$$

$$= tt+cc-\mathrm{TT} = tt-\frac{ff}{tt}xx = \frac{tt}{cc}yy+\frac{cc}{tt}xx = \frac{\mathrm{DC}}{\mathrm{DT}}\times\overline{\mathrm{PT}}^2.$$

$$\overline{\mathrm{CF}}^2 = ff = tt-cc = tt-tp = \frac{cc}{p}\times\overline{t-p} = \frac{tt}{xx}\times mm = \mathrm{P}\varrho\times\mathrm{T}t$$

$$= \frac{tt}{\mathrm{CC}}\times\mathrm{FM}f = \frac{cc}{\mathrm{CC}}\times\overline{fm}^2 = \frac{mm}{\mathrm{CC}}\times\mathrm{FT}f = \frac{mm}{\mathrm{CC}}\times\mathrm{NT}n.$$

The ſemi-parameter (p) to the greater axis ($\mathrm{A}a$) is equal to

$$\mathrm{F}\Phi = \mathrm{PZ} = \frac{\mathrm{AF}a}{\mathrm{AC}} = \frac{\mathrm{V}a\mathrm{U}}{\mathrm{AC}} = \frac{\mathrm{CFG}}{\mathrm{AC}} = \frac{t\mathrm{C}d}{\mathrm{AC}} = \frac{\mathrm{PF}\Pi}{\frac{1}{2}\mathrm{P}\Pi} = \frac{\mathrm{PM}\times\mathrm{C}\varrho}{\mathrm{AC}}$$

$$= \frac{\mathrm{FR}\times fr}{\mathrm{AC}} = \frac{\mathrm{AN}\times an}{\mathrm{AC}} = \frac{\mathrm{AC}\times\mathrm{PM}}{\mathrm{P}m} = \frac{\mathrm{AC}\times\mathrm{DM}}{\mathrm{DC}} = \frac{\mathrm{AC}\times d\mathrm{C}}{dm} = \frac{\mathrm{BC}\times\mathrm{PM}}{\mathrm{CQ}}$$

$$= \frac{\mathrm{BC}\times\mathrm{CQ}}{\mathrm{P}m} = \frac{\overline{\mathrm{BC}}^2}{\mathrm{AC}} = \frac{cc}{t} = \frac{ttyy}{sx} = \frac{\pm tt\mp ff}{t} = \frac{cc-yy}{xx}\times t$$

$$= \frac{\mathrm{CC}+\mathrm{TT}-tt}{t} = \frac{\mathrm{CC}+xx-tt}{xx}\times t = \frac{2t-z}{tz}\times\overline{\mathrm{FR}}^2 = \frac{2t-v}{tv}\times\overline{fr}^2.$$

The ſemi-parameter (P) to any diameter ($\mathrm{P}p$) is equal to

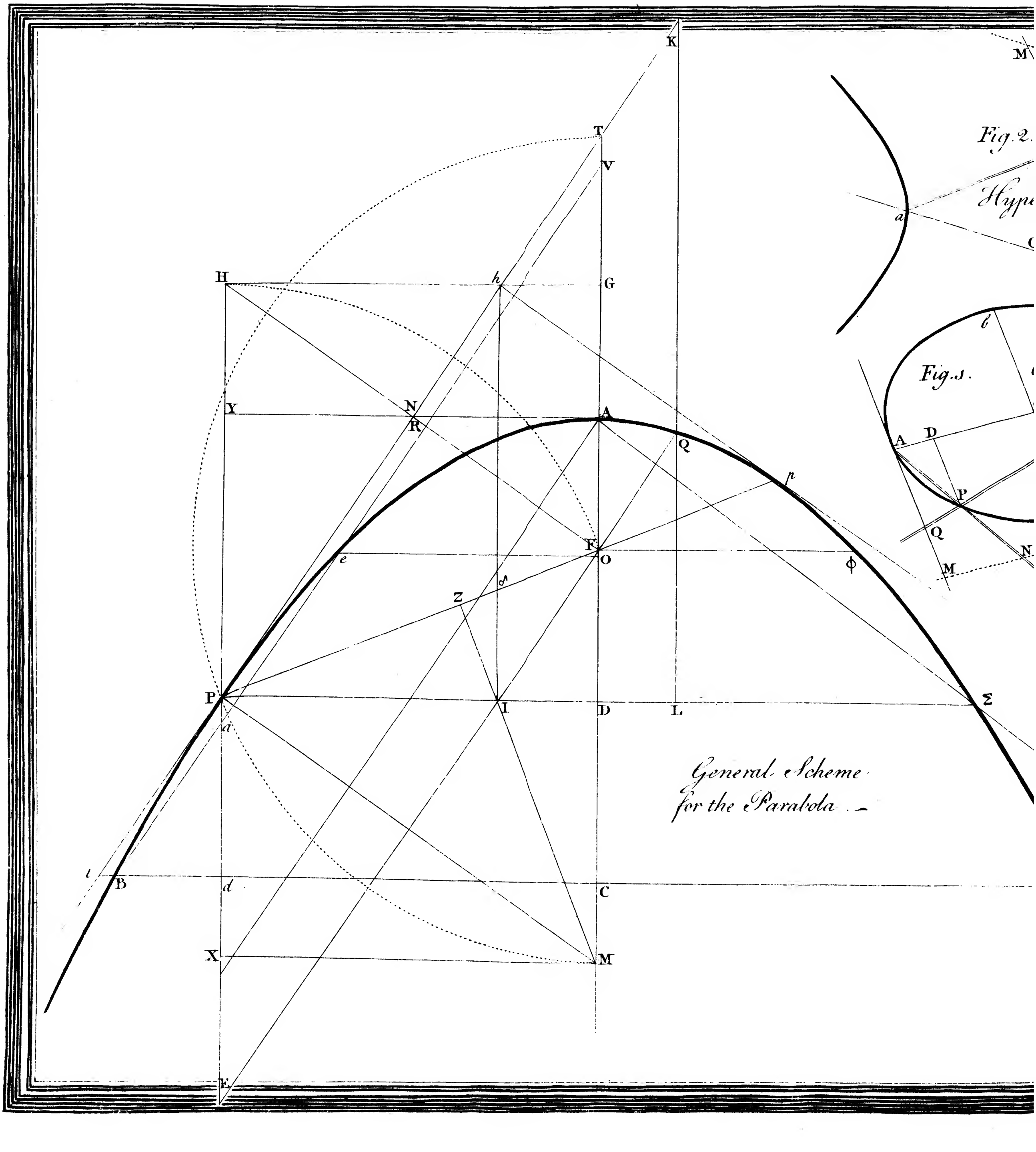
$$\mathrm{P}\eta = \frac{\overline{\mathrm{CQ}}^2}{\mathrm{PC}} = \frac{\mathrm{CC}}{\mathrm{T}} = \frac{\mathrm{MP}m}{\mathrm{PC}} = \frac{\mathrm{NP}n}{\mathrm{PC}} = \frac{\mathrm{TP}t}{\mathrm{PC}} = \frac{\mathrm{FP}f}{\mathrm{PC}} = \frac{\mathrm{L}pl}{\mathrm{PC}} = \frac{\mathrm{PN}\times p\mathrm{L}}{\mathrm{PC}}$$

$$= \frac{\mathrm{CD}\times\mathrm{TM}}{\mathrm{PC}} = \frac{\mathrm{PD}\times tm}{\mathrm{PC}} = \frac{\mathrm{PT}\times\mathrm{PN}}{\mathrm{PV}} = \frac{tt}{cc}\times\frac{\overline{\mathrm{PM}}^2}{\mathrm{PC}} = \frac{cc}{tt}\times\frac{\mathrm{P}m^2}{\mathrm{PC}}$$

$$= \frac{cc}{ff}\times\frac{\overline{fm}^2}{\mathrm{PC}} = \frac{tt}{ff}\times\frac{\mathrm{PM}f}{\mathrm{PC}} = \frac{tt+cc-\mathrm{TT}}{\mathrm{T}}.$$

[Sce Tab. XIV, XV.]

XXXVII. *An*



General Scheme for the Parabola.

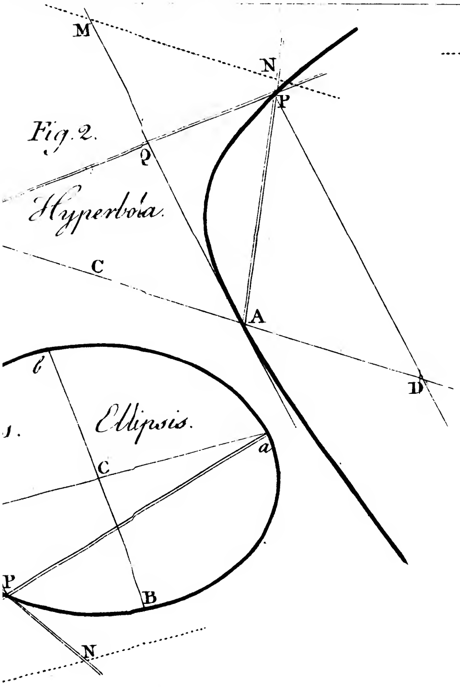


Fig. 2.
Hyperbola.

1.
Ellipsis.

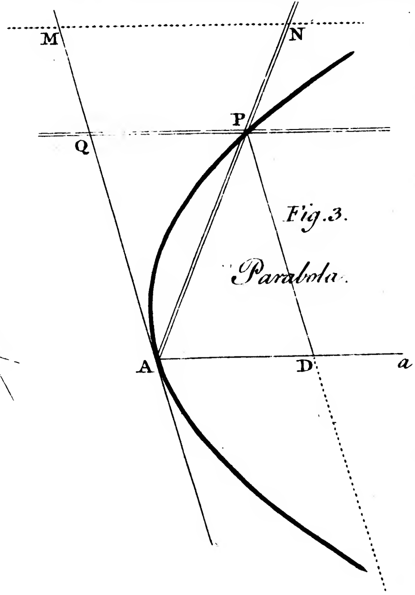


Fig. 3.
Parabola.

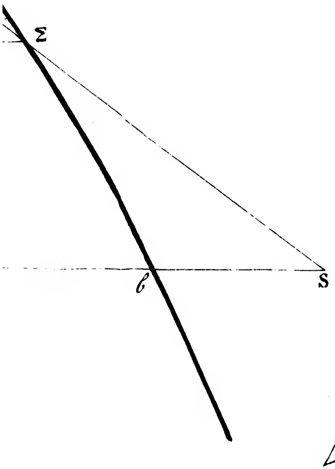
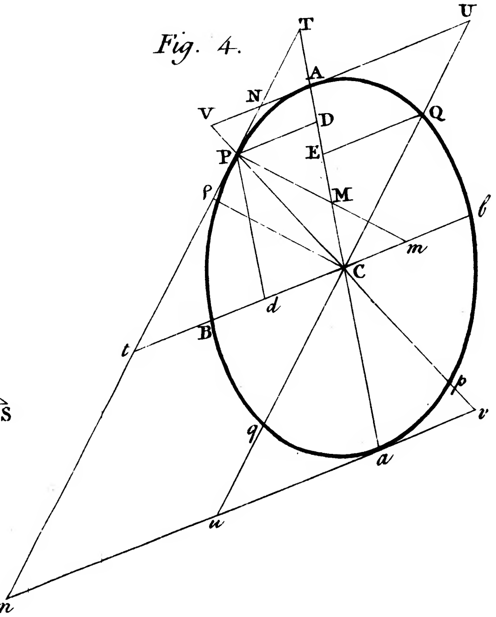
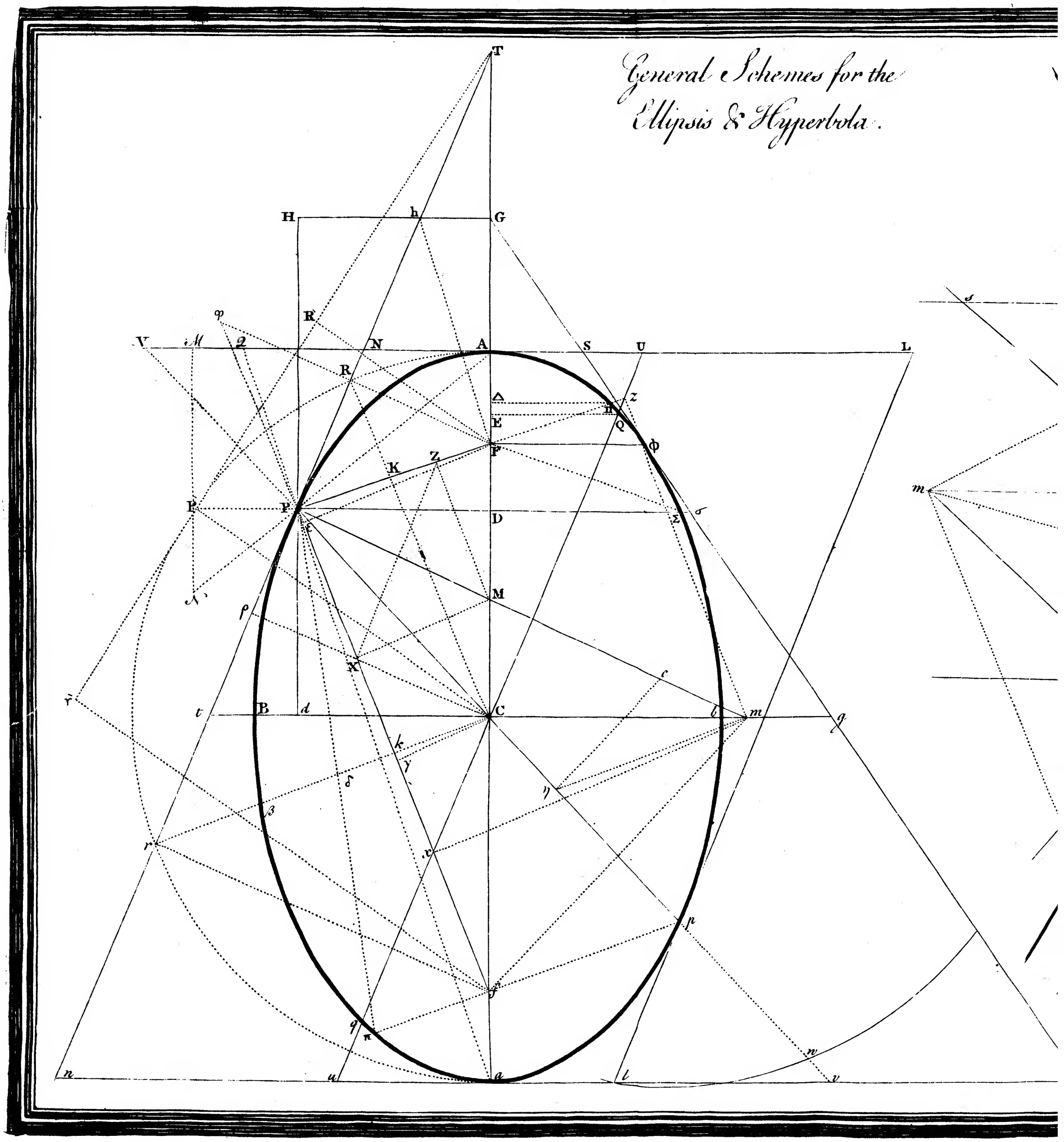


Fig. 4.


Basire sc.

General Schemes for the Ellipsis & Hyperbola.



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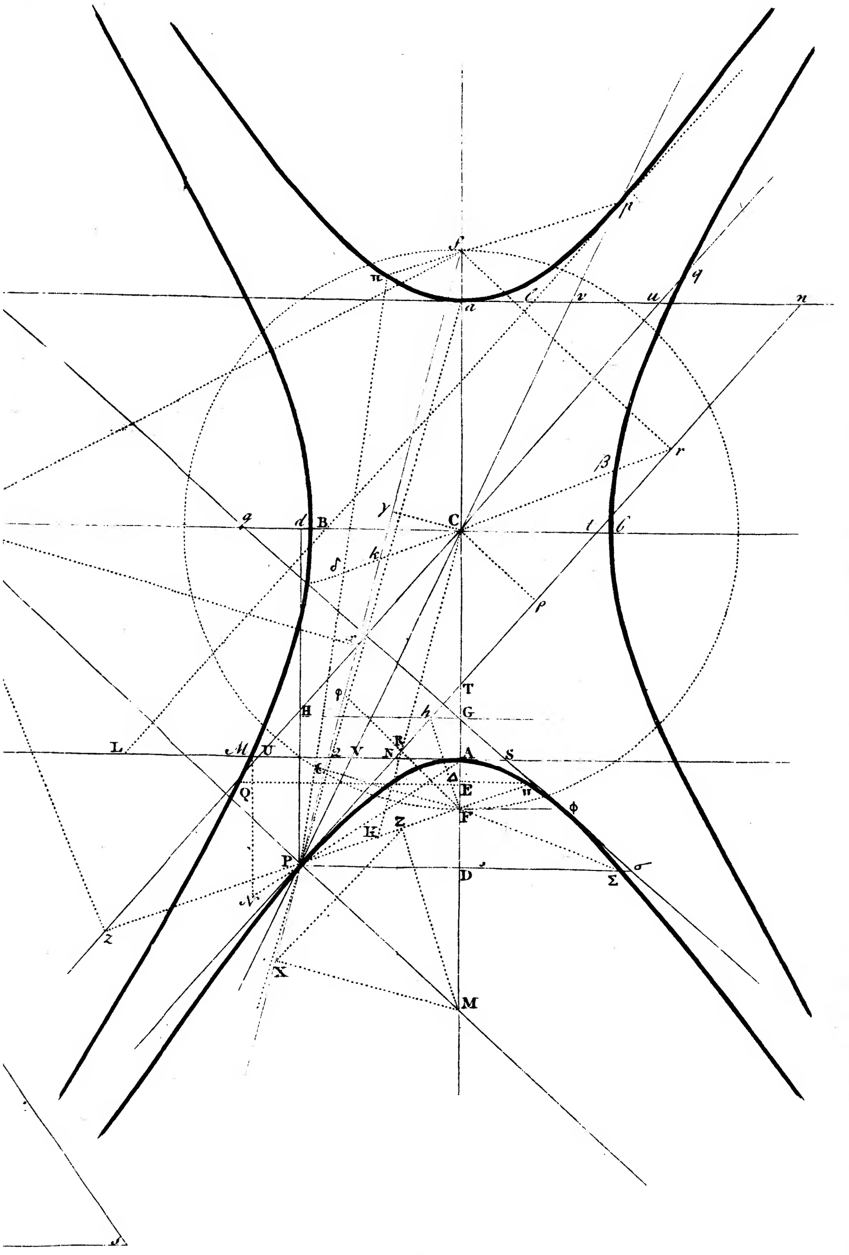
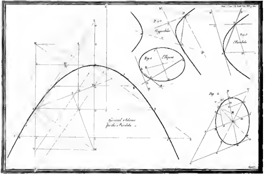
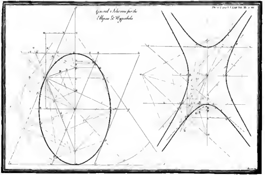


Basire sc.


General Scheme for the Parabola
Fig. 1
Hyperbola
Fig. 2
Ellipse
Fig. 3
Parabola
Fig. 4


General Solutions for the
Ellipsis & Hyperbola